NAME: Solutions **STUDENT #:**

- There is a total of 42 marks; the maximum grade is 40 (2 bonus marks)
- Check that you have a total of 6 distinct pages and notify your TA if this is not the case
- Calculators are not allowed
- Phones and other devices should be turned off and hidden
- Have your student card face up on your desk
- You have to show all your work for all the questions, except the True/False and multiple choice

Question 1. Find the cofactor C_{32} of the matrix $\begin{bmatrix} 1 & 2 & -2 \\ 3 & 5 & 1 \\ 7 & -1 & 3 \end{bmatrix}$. [2 marks]

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} = -(1+5) = -6$$

Question 2. You are told that a 3×3 matrix A has matrix of cofactors $\begin{bmatrix} 1 & 2 & -2 \\ 3 & 5 & 1 \\ 7 & -1 & 3 \end{bmatrix}$, and $\det(A) = 10$.

a. Explain why A is invertible. [1 mark]

b. Find the adjoint $\mathrm{Adj}(A)$ of A. [1 mark]

c. Find A^{-1}. [2 marks]

a) $\det(A) \neq 0$ so A is invertible

b) $\mathrm{Adj}(A) = \begin{bmatrix} 1 & 2 & -2 \\ 3 & 5 & 1 \\ 7 & -1 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 5 & -1 \\ -2 & 1 & 3 \end{bmatrix}$

c) $A^{-1} = \dfrac{1}{\det(A)} \mathrm{Adj}(A) = \dfrac{1}{10} \begin{bmatrix} 1 & 3 & 7 \\ 2 & 5 & -1 \\ -2 & 1 & 3 \end{bmatrix}$

Question 3. Consider the system

$$\begin{aligned} 2x + y + z &= 1 \\ y + 4z &= 0 \\ x + 2y - 2z &= 3 \end{aligned}$$

a. Explain why Cramer's rule can be applied to solve this system. [2 marks]

b. Use Cramer's rule in order to find the component y of the solution (x, y, z) of this system. Methods other than Cramer's rule will not be accepted. [2 marks]

a) $\begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 4 \\ 1 & 2 & -2 \end{vmatrix} = 2\begin{vmatrix} 1 & 4 \\ 2 & -2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} =$

$= 2(-2-8) + (4-1) = -20+3 = -17 \neq 0$

Since the determinant of the coefficient matrix is nonzero, we can use Cramer's rule.

b) $$y = \frac{\begin{vmatrix} 2 & \cancel{1} & 1 \\ \cancel{0} & \cancel{0} & \cancel{4} \\ 1 & \cancel{3} & -2 \end{vmatrix}}{-17}$$

$= \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -4-1 = -5$

$$= \frac{-5 \cdot 4}{-17} = \frac{20}{17}$$

Question 4. Can $\begin{bmatrix} 5 \\ -4 \\ 15 \end{bmatrix}$ be written as a linear combination of $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$? If yes, give such a linear combination. If no, explain why this is the case. [4 marks]

We solve

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ -2 & 1 & -4 & -4 \\ 3 & 0 & 6 & 15 \end{array}\right] \xrightarrow[R_2' = R_2 + 2R_1]{R_3' = R_3 - 3R_1} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 2 & 5 \\ 0 & \boxed{1} & 0 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

∞ solutions: $\begin{aligned} x &= 5 + 2t \\ y &= 6 \\ z &= t \end{aligned}$, $t \in \mathbb{R}$. for $t = 0$: $\begin{aligned} x &= 5 \\ y &= 6 \\ z &= 0 \end{aligned}$

So

$$\begin{bmatrix} 5 \\ -4 \\ 15 \end{bmatrix} = 5\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$$

Question 5. Let $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. Does S span $\mathbb{R}^3$? Justify your answer. [4 marks]

We check if the system below is consistent for all $a, b, c \in \mathbb{R}$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 1 & 1 & 1 & b \\ 2 & 0 & 0 & c \end{array}\right] \xrightarrow[R_3' = R_3 - 2R_1]{R_2' = R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 0 & b-a \\ 0 & 0 & -2 & c-2a \end{array}\right]$$

This is consistent for all $a, b, c \in \mathbb{R}$, thus S spans $\mathbb{R}^3$.

Question 6. Prove that S_1 is a subspace of $\mathbb{R}^3$ and that S_2 is not. [8 marks]

$$S_1 = \left\{ \begin{bmatrix} a+b \\ b \\ 2a \end{bmatrix} : a, b \in \mathbb{R} \right\}, \quad S_2 = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : b = 2a+2 \right\}$$

S_1

- S_1 is not empty.
- we check S_1 is closed under addition:

Let $\vec{u}, \vec{v} \in S_1$. Then $\vec{u} = \begin{bmatrix} a_1+b_1 \\ b_1 \\ 2a_1 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} a_2+b_2 \\ b_2 \\ 2a_2 \end{bmatrix}$

we have

$$\vec{u}+\vec{v} = \begin{bmatrix} a_1+b_1+a_2+b_2 \\ b_1+b_2 \\ 2a_1+2a_2 \end{bmatrix} = \begin{bmatrix} (a_1+a_2)+(b_1+b_2) \\ (b_1+b_2) \\ 2(a_1+a_2) \end{bmatrix} \in S_1$$

So S_1 is closed under addition.

- Let $\vec{u} = \begin{bmatrix} a+b \\ b \\ 2a \end{bmatrix} \in S$ and $r \in \mathbb{R}$.

Then $r\vec{u} = \begin{bmatrix} r(a+b) \\ rb \\ r(2a) \end{bmatrix} = \begin{bmatrix} (ra)+(rb) \\ rb \\ 2(ra) \end{bmatrix} \in S$

So S_1 is closed under scalar multiplication.

S_2

We observe that $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin S_2$ because the third component of $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not equal to $2 \cdot 0 + 2 = 2$. So S_2 is not a subspace of $\mathbb{R}^3$.

True/False and multiple choice questions

Question 7. Circle T for *True* and F for *False*. Do not justify your answers, just circle your choice. [2 marks each]

1. The sum of two vectors of $\mathbb{R}^2$ is a vector of $\mathbb{R}^4$. **T** **F**

2. Let S be a spanning set of a vector space V. Then every vector from V can be written as a linear combination of elements from S always in *only one* way. **T** **F**

3. If the columns of a square matrix are linearly independent, then it is invertible. **T** **F**

4. $\begin{bmatrix} 0 & 0 \\ 5 & 2 \end{bmatrix}$ belongs in $\mathrm{span}\left(\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\right\}\right)$ **T** **F**

Question 8. In the following do not justify your answer, just circle one letter. [2 marks each]

1. The unit vector in the direction of $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is:

$\sqrt{1^2+2^2+(-1)^2} = \sqrt{6}$

A $\begin{bmatrix} 1/6 \\ 2/6 \\ -1/6 \end{bmatrix}$ **B** $\begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$ **C** $\begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix}$ **D** $\begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$ **E** $\begin{bmatrix} -1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

2. For which k are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ k \end{bmatrix}$ linearly independent?

$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & k \end{vmatrix} = k-1$

LI: $k-1 \neq 0$

A Only for $k = 1$

B For all $k \in \mathbb{R}$ with $k \neq 1$

C For all $k \in \mathbb{R}$

D Only for $k = 0$

E For all $k \in \mathbb{R}$ with $k \neq 0$

Question 9. Circle **LI** for *linearly independent*, and **LD** for *linearly dependent*, to describe the following sets. Do not justify your answers. [2 marks each]

$\left\{\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 10 & 6 \\ 0 & 6 \end{bmatrix}\right\}$ **LI** **LD**

not multiples of each other

$\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 10 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 4 \\ -1 \end{bmatrix}\right\}$ **LI** **LD**

5 vectors of $\mathbb{R}^4$

NAME: Solutions **STUDENT #:**

- There is a total of 42 marks; the maximum grade is 40 (2 bonus marks)
- Check that you have a total of 6 distinct pages and notify your TA if this is not the case
- Calculators are not allowed
- Phones and other devices should be turned off and hidden
- Have your student card face up on your desk
- You have to show all your work for all the questions, except the True/False and multiple choice

Question 1. Find the cofactor C_{13} of the matrix $\begin{bmatrix} 1 & 2 & 2 \\ 3 & 5 & 1 \\ 7 & -1 & 3 \end{bmatrix}$. [2 marks]

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 3 & 5 \\ 7 & -1 \end{vmatrix} = -3 - 5 \cdot 7 = -38$$

Question 2. Can $\begin{bmatrix} 7 \\ -17 \\ 14 \end{bmatrix}$ be written as a linear combination of $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ -9 \\ 6 \end{bmatrix}$? If yes, give such a linear combination. If no, explain why this is the case. [4 marks]

We solve:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ -3 & 1 & -9 & -17 \\ 2 & 0 & 6 & 14 \end{array}\right] \xrightarrow[R_3' = R_3 - 2R_1]{R_2' = R_2 + 3R_1} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 3 & 7 \\ 0 & \boxed{1} & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

∞ Solutions: $\begin{matrix} x = 7 - 3t \\ y = 4 \\ z = t \end{matrix}$, $t \in \mathbb{R}$. For $t = 0$ $\begin{matrix} x = 7 \\ y = 4 \\ z = 0 \end{matrix}$

So

$$\begin{bmatrix} 7 \\ -17 \\ 14 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ -9 \\ 6 \end{bmatrix}$$

Question 3. Consider the system

$$\begin{aligned} 2x + y + z &= 1 \\ x + 2y - 2z &= 3 \\ y + 4z &= 0 \end{aligned}$$

a. Explain why Cramer's rule can be applied to solve this system. [2 marks]

b. Use Cramer's rule in order to find the component z of the solution (x, y, z) of this system. Methods other than Cramer's rule will not be accepted. [2 marks]

a) $\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & -2 \\ 0 & 1 & 4 \end{vmatrix} = 2\begin{vmatrix} 2 & -2 \\ 1 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}$

$= 2(8+2) - (4-1)$

$= 20 - 3 = 17 \neq 0$

Since the determinant of the coefficient matrix is nonzero, we can use Cramer's rule

b) $z = \dfrac{\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 0 \end{vmatrix}}{17} = -\dfrac{5}{17}$

Question 4. You are told that a 3×3 matrix A has matrix of cofactors $\begin{bmatrix} 13 & 4 & 12 \\ 7 & 3 & 9 \\ -16 & -10 & -19 \end{bmatrix}$, and $\det(A) = -11$.

a. Explain why A is invertible. [1 mark]

b. Find the adjoint $\mathrm{Adj}(A)$ of A. [1 mark]

c. Find A^{-1}. [2 marks]

a) $\det(A) = -11 \neq 0$, so A is invertible

b) $$\mathrm{Adj}(A) = \begin{bmatrix} 13 & 4 & 12 \\ 7 & 3 & 9 \\ -16 & -10 & -19 \end{bmatrix}^T = \begin{bmatrix} 13 & 7 & -16 \\ 4 & 3 & -10 \\ 12 & 9 & -19 \end{bmatrix}$$

c) $$A^{-1} = \frac{1}{\det(A)} \mathrm{Adj}(A)$$

$$= \frac{1}{-11} \begin{bmatrix} 13 & 7 & -16 \\ 4 & 3 & -10 \\ 12 & 9 & -19 \end{bmatrix}$$

Question 5. Let $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$. Does S span $\mathbb{R}^3$? Justify your answer. [4 marks]

We check if the system below is consistent for all $a, b, c \in \mathbb{R}$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 2 & 0 & 0 & b \\ 0 & 2 & 2 & c \end{array}\right] \xrightarrow{R_2' = R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & -2 & 0 & b-2a \\ 0 & 2 & 2 & c \end{array}\right]$$

$$\xrightarrow{R_3' = R_3' + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & -2 & 0 & b-2a \\ 0 & 0 & 2 & c+b-2a \end{array}\right]$$

This is consistent for all $a, b, c \in \mathbb{R}$, thus S spans $\mathbb{R}^3$.

Question 6. For each of the following subsets of $\mathbb{R}^3$, prove whether they are subspaces of $\mathbb{R}^3$ or not. [8 marks]

$$S_1 = \left\{ \begin{bmatrix} 0 \\ a \\ b \end{bmatrix} : b = 3a - 3 \right\}, \quad S_2 = \left\{ \begin{bmatrix} b-a \\ 2b \\ a \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

S_2

- S_2 is not empty
- We check S_1 is closed under addition:

Let $\vec{u}, \vec{v} \in S_2$. Then $\vec{u} = \begin{bmatrix} b_1 - a_1 \\ 2b_1 \\ a_1 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} b_2 - a_2 \\ 2b_2 \\ a_2 \end{bmatrix}$.

We have
$\vec{u} + \vec{v} = \begin{bmatrix} b_1 - a_1 + b_2 - a_2 \\ 2b_1 + 2b_2 \\ a_1 + a_2 \end{bmatrix} = \begin{bmatrix} (b_1+b_2) - (a_1+a_2) \\ 2(b_1+b_2) \\ (a_1+a_2) \end{bmatrix} \in S_2$

So S_2 is closed under addition

- Let $\vec{u} = \begin{bmatrix} b-a \\ 2b \\ a \end{bmatrix} \in S$ and $r \in \mathbb{R}$.

So S_2 is closed under scalar multiplication.

S_1

We observe that $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin S_1$ because the third component of $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not equal to $3 \cdot 0 - 3 = -3$. So S_1 is not a subspace of $\mathbb{R}^3$.

True/False and multiple choice questions

Question 7. Circle T for *True* and F for *False*. Do not justify your answers, just circle your choice. [2 marks each]

1. $\begin{bmatrix} 0 & 7 \\ 0 & 2 \end{bmatrix}$ does not belong in $\operatorname{span}\left(\left\{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\right\}\right)$. **T** **F**

2. If the columns of a square matrix are linearly dependent then its determinant is not zero. **T** **F**

3. Suppose that S is a spanning set of a vector space V. Then there is always *only one* way of writing every vector from V as a linear combination of elements from S. **T** **F**

4. The addition of two vectors from $\mathbb{R}^3$ gives a vector of $\mathbb{R}^6$. **T** **F**

Question 8. In the following do not justify your answer, just circle one letter. [2 marks each]

1. The unit vector in the direction of $\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$ is:

A $\begin{bmatrix} 3/9 \\ 1/9 \\ -1/9 \end{bmatrix}$ **B** $\begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}$ **C** $\begin{bmatrix} -3/\sqrt{11} \\ -1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}$ **A** $\begin{bmatrix} 3/11 \\ 1/11 \\ -1/11 \end{bmatrix}$ **E** $\begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ -1/\sqrt{11} \end{bmatrix}$

2. For which k are the vectors $\begin{bmatrix} 1 \\ 1 \\ k \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ linearly dependent?

A For all $k \in \mathbb{R}$ with $k \neq 0$

B For all $k \in \mathbb{R}$ with $k \neq 1$

C Only for $k = 1$

D Only for $k = 0$

E For all $k \in \mathbb{R}$

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ k & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ k & 1 \end{vmatrix} = -(1-k) = k-1$$

LD $\Rightarrow$ $k-1=0$

Question 9. Circle **LI** for *linearly independent*, and **LD** for *linearly dependent*, to describe the following sets. Do not justify your answers. [2 marks each]

$\left\{\begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 5 \end{bmatrix}\right\}$ **LI** **LD**

first + second

$\left\{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}\right\}$ **LI** **LD**

contains the zero vector